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Mode Traces in Degenerate Eigensystems and Augmented Assurance

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Eigenpairs, contextually indicative of the frequencies and oscillatory modes of structural systems, are considered as functions of a single parameter. Undesired permutation of modal designations is noted to arise following events of transitory eigenvalue coalescence with respect to the parameter, resulting in the definition of nonsmooth functions. The tracing of eigenpairs across such events is outlined for the permanently degenerate, general eigenproblem as the general case; the cases of transitorily degenerate and distinct eigenvalues are thus accounted for. The foundation for mode tracing is the assumption of eigenvector consistency across parameter intervals, used as a means of eigenpair reconciliation. The resulting traced modes are smooth and their variations physically pertinent, which is necessary in iterative schemes in which convergence may otherwise be jeopardized. The proposal of an augmented modal assurance routine that potentially augments the assurance of tracing and, therefore, the maximum permissible parameter perturbation, is given; a threefold increase in the insight to consistency entails. The notion of the routine is to forward and backward cast eigenvectors utilizing their derivatives in affine modeling. A numerical example of the modes of a cyclic frame as functions of a cyclic distribution of membrane forces demonstrates the concepts and utility of the proposal.

Nomenclature

$\boldsymbol{A}, \boldsymbol{B}$	=	$n \times n$ square matrices
$\mathrm{diag}(\boldsymbol{a}_i)$	=	diagonal matrix formed by elements a_i of vector $\boldsymbol{a}$
$\boldsymbol{I}$	=	identity matrix
i	=	matrix row designation
j	=	matrix column designation
k	=	iteration number
$\boldsymbol{M}$	=	eigenvector consistency matrix
m	=	degree of eigenvalue degeneracy, multiplicity
n	=	order of eigenproblem
r	=	independent variable
T	=	matrix transpose
tr	=	matrix trace
$\boldsymbol{X}$	=	matrix of m eigenvectors associated with degenerate eigenvalue
$\boldsymbol{x}$	=	eigenvector
$\boldsymbol{Z}$	=	$\boldsymbol{X}$ transformed to orthogonal set
$\overline{\boldsymbol{Z}^{k+1}}$	=	$\boldsymbol{Z}$ with eigenvector bases aligned to those at preceding iterate
z_d	=	dth eigenvector of matrix $\boldsymbol{z}$
$\boldsymbol{\Gamma}$	=	$m \times m$ transformation matrix orthogonalizing $\boldsymbol{X}$
$\bar{\boldsymbol{\Gamma}}$	=	$m \times m$ transformation matrix aligning vector bases
δ	=	Kronecker delta
$\boldsymbol{\Theta}$	=	Boolean operator
$\boldsymbol{\Lambda}$	=	diagonal matrix of degenerate eigenvalues
λ	=	eigenvalue
λ_d	=	dth eigenvalue
$\vert\ \vert$	=	modulus
$\{\ \}$	=	set, vector set
$\Vert\ \Vert_2$	=	Euclidean norm
$\Vert\ \Vert_\infty$	=	infinity norm
$\to$	=	set permutation

Introduction

IN the general eigenproblem dependent on a parameter, where the eigenpairs are functions of that parameter, the designation of the eigenpairs is arbitrary. Such problems occur in structural optimization and system identification, the eigenvalues and eigenvectors being, respectively, indicative of resonance frequencies and modes of oscillation. Ordering in terms of ascending or descending eigenvalues entails erroneous permutations across a parameterwise interval of eigenvalue coalescence, the intersection of eigenvalues at particular parameter values, occurs within the interval. Such an ordering of modes may be deemed a classification in the sense of modal energies and, consequently, undesired effects arise. Figure 1 illustrates the nature of these discontinuous functions and how they arise. Affected are schemes that iteratively converge to sought parameter values. In optimization, evaluations of the objective and constraint functions become inconsistent between the initial configuration and the converged solution. In both optimization and identification schemes utilizing eigenvalue gradients in search algorithms, the discontinuous, energy-classified modes present nonsmooth eigenvalue functions of the parameter and, thus, convergence may be prevented altogether. This is the result of gradient methods relying on function smoothness. An example is Newton's method, in which the function value and derivative are used to make an affine model of the function to evaluate an excursion to the proceeding iterate, hopefully progressing to the problem root. Mode permutation at subsequent iterates disrupts the iterative scheme through abrupt changes in the gradients of the nonsmooth functions.

It is intuitive in the context of structural dynamics to assume that the eigenvectors vary significantly less than their eigenvalues with respect to a particular parameter. This has been previously exploited in correlating modes preceding and proceeding parameter perturbation. The modal assurance criterion[1] (MAC) has thus been used to develop a Boolean operator, operating on the nominated subset of eigenpairs of a current iterate and permuting them to concur in a physical sense with those of the preceding iterate.[2] It is, thus, possible to distinguish the physically pertinent smooth functions, such as are discernible from Fig. 1 through deduction, from the nonsmooth functions in an automated sense. The same assurance criterion proved successful in application to tracing modes in structural topology optimization.[3] In an analogous manner, cross mass

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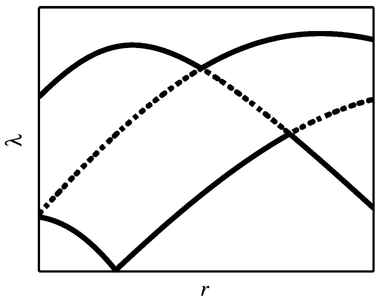

Fig. 1 Energy classification of eigenvalues: - - - -, discontinuous functions.

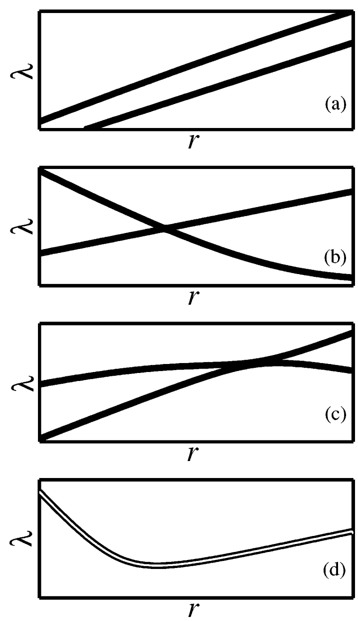

Fig. 2 Degrees of degeneracy: a) distinctness, b) first-order transitory, c) second-order transitory, and d) permanent.

orthogonality (XOR) has been implemented to the same end with good effect.[4,5] The aforementioned methods of mode tracing are advantageous in their computational efficiency compared to methods dealing with perturbation expansions of the eigenproblem,[6] although assurance of correlation may be wanting if parameter intervals are large. This may demand a compromise of the iterative excursion length to ensure adequate modal assurance. Presently, a means of augmenting this assurance is proposed, that such a compromise may not have to be made. The proposal concerns the forward and backward casting of the eigenvector freedoms via knowledge of their derivatives and the parameter perturbation. Thus, the computational expense of heightening assurance by solving the eigenproblem at parameter values progressively closer to the current iterate may be avoided.

The degenerate case, which has the physical interpretation of repeated frequencies, is treated as a generalized case of mode tracing, with nondegenerate cases treated inclusively. Both localized and permanent degeneracies of eigenvalues are considered: the former relates to the transitory coalescence of eigenvalues and the latter to eigenvalues that are mutually degenerate for all values of the parameter. Indeed, the transitory case can be of varying order, as is illustrated by Fig. 2. The interception of eigenvalue functions in a first-order coalescence is associated with degenerate eigenvalues, with all of the eigenvalue derivatives being distinct; this is shown in Fig. 2b. In second-order coalescence, both the eigenvalues and the eigenvalue first derivatives are degenerate, with the remaining derivatives being distinct. This is shown in Fig. 2c. Thus, a qth-order coalescence is defined as one in which the qth and subsequent derivatives are distinct. Such instances are presently deemed unlikely, with only the first- and infinitieth-order (permanent) instances being commonly encountered, and are, therefore, not considered here. Treatment of the second-order eigenvalue coalescence may be found in the literature.[7,8]

Particular treatment in aligning the eigenvector bases to those bases at some perturbed parameter value is required because if there is degeneracy of eigenvalues, these vector bases will be arbitrary and nonunique. Once aligned, continuity of the eigenvector freedoms with respect to a parameter and, therefore, differentiability, can be observed. Furthermore, correct evaluation of vector consistency can only be made thereafter. The alignment of bases is discussed for different degrees of degeneracy. These degrees also affect the way in which the eigenvector derivatives, needed for the forward and backward vector casting, are obtained.

Theory

The symmetric, general eigenproblem is considered:

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{B}\boldsymbol{x} \tag{1}$$

in which it is assumed that the eigenvectors are normalized as

$$\boldsymbol{x}^T \boldsymbol{B}\boldsymbol{x} = 1 \tag{2}$$

and that $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{x}$, and λ are functions of a single independent variable r. At degeneracy, Eqs. (1) and (2) may be expressed, respectively, as

$$\boldsymbol{A}\boldsymbol{X} = \boldsymbol{B}\boldsymbol{X}\boldsymbol{\Lambda} \tag{3}$$

$$\boldsymbol{X}^T \boldsymbol{B}\boldsymbol{X} = \boldsymbol{I} + \boldsymbol{N} \tag{4}$$

Degenerate eigenvalues are associated with eigenvectors that are arbitrary and non-unique. Furthermore, they will not necessarily be orthogonal to one another, so that the right-hand side of Eq. (4) is not purely $\boldsymbol{I}$ but involves nonzero, nondiagonal terms $\boldsymbol{N}$.

The proposal potentially augmenting the assurance with which eigenpairs can be correlated across parameter intervals is later outlined. It is first necessary to define the derivatives of the eigenpairs, and, as shall be seen, this is dependent on the degree of degeneracy. The method of evaluating eigenvector derivatives due to Nelson[9] was extended by Ojalvo[10] to account for eigenvalue multiplicity. The method of Nelson is appealing in that the bandedness of the matrices of Eq. (1) is preserved and, hence, efficient solution of the eigenproblem is allowed. Additionally, only that eigenpair associated with the derivative is required in the computation.

Under the assumption that there exist eigenvectors $\boldsymbol{Z}$ that form an orthogonal set, defining Eq. (3) for these vectors and differentiating leads to

$$(\boldsymbol{A} - \lambda\boldsymbol{B})\frac{\partial \boldsymbol{Z}}{\partial r} = \boldsymbol{B}\boldsymbol{Z}\frac{\partial \boldsymbol{\Lambda}}{\partial r} - \left(\frac{\partial \boldsymbol{A}}{\partial r} - \lambda\frac{\partial \boldsymbol{B}}{\partial r}\right)\boldsymbol{Z} \tag{5}$$

From Eq. (5), Ojalvo derives the mth-order auxiliary eigenproblem relating to the m-fold degenerate eigenvalues:

$$\left[\boldsymbol{X}^T\left(\frac{\partial \boldsymbol{A}}{\partial r} - \lambda\frac{\partial \boldsymbol{B}}{\partial r}\right)\boldsymbol{X}\right]\boldsymbol{\Gamma} = \boldsymbol{\Gamma}\left(\frac{\partial \boldsymbol{\Lambda}}{\partial r}\right) \tag{6}$$

The eigenvalue derivatives and the $m \times m$ transformation matrix $\boldsymbol{\Gamma}$ are then obtained as the eigenpairs of Eq. (6). The latter achieves orthogonality of the set $\boldsymbol{X}$ of eigenvectors associated with a degenerate eigenvalue through

$$\boldsymbol{Z} = \boldsymbol{X}\boldsymbol{\Gamma}\left[\mathrm{diag}\left(w_i^{-\frac{1}{2}}\right)\right], \qquad i = 1, \ldots, m \tag{7}$$

where

$$\boldsymbol{w} = \begin{Bmatrix} \left(\phi_1^T \boldsymbol{B}\phi_1\right) \\ \vdots \\ \left(\phi_m^T \boldsymbol{B}\phi_m\right) \end{Bmatrix} \tag{8}$$

with $\boldsymbol{\Phi} \equiv \boldsymbol{X}\boldsymbol{\Gamma}$. In the structural dynamics context, the vector $\boldsymbol{w}$ is a vector of modal masses; postmultiplication of Eq. (7) by the diagonal matrix restores the $\boldsymbol{B}$-normality of eigenvectors. It now holds that

$$Z^T B Z = I \tag{9}$$

Consistently, the eigenvalue derivatives arising from the solution to the eigenproblem (6) require weighting,

$$\operatorname{diag}\left(\frac{\partial \lambda}{\partial r}\right) = \frac{\partial \boldsymbol{\Lambda}}{\partial r} \operatorname{diag}\left(w_i^{-1}\right) \tag{10}$$

All eigenvectors will henceforth be denoted by z, corresponding to those of degenerate eigenvalues transformed or else to distinct eigenvalues, so that the set of vectors $\{z_1, \ldots, z_n\}$ forms an entirely orthonormal set.

Enforcing orthogonality by Eq. (7) gives rise to the arbitrary, nonunique eigenvector bases at a point of first-order transitory eigenvalue coalescence aligning with those unique bases adjacent to coalescence, where the eigenvalues have separated and become distinct. Thus, continuity of the eigenvector freedoms and, hence, their differentiability, can be observed. In the case of permanent eigenvalue degeneracy, however, the eigenvector bases are arbitrary and nonunique for all r. These eigenvectors are nevertheless of use in the present mode tracing task, as much as are the unique eigenvectors, but it remains to align the bases across r-wise intervals. When the permanently degenerate mode eigenvectors at the initial solution to Eq. (1), once orthonormalized by Eq. (7), are considered as the datum set, subsequent bases can be aligned to these by defining the transformation

$$\bar{\boldsymbol{\Gamma}} = [(Z^{k+1})^T Z^{k+1}]^{-1} [(Z^{k+1})^T Z^k] \tag{11}$$

which serves to perform the least-squares minimization

$$\min\left(\operatorname{tr}[(Z^{k+1}\bar{\boldsymbol{\Gamma}} - Z^k)^T (Z^{k+1}\bar{\boldsymbol{\Gamma}} - Z^k)]\right) \tag{12}$$

whence the aligned eigenvectors at the proceeding iterate are given as

$$\overline{Z^{k+1}} = Z^{k+1}\bar{\boldsymbol{\Gamma}}\left[\operatorname{diag}\left(w_i^{-\frac{1}{2}}\right)\right], \qquad i = 1, \ldots, m \tag{13}$$

The vector $\boldsymbol{w}$ is defined by Eq. (8), with $\boldsymbol{\Phi} \equiv Z^{k+1}\bar{\boldsymbol{\Gamma}}$, and serves once more to preserve $\boldsymbol{B}$-normality. Such vector alignment has been used in the reconciliation of mathematical and observed oscillatory modes in the presence of repeated frequencies.[11,12] Presently, it is used in ensuring continuity and differentiability of the eigenvector freedoms, as shall now be detailed.

Mills-Curran[13] and Dailey[14] present the method by which the eigenvector derivatives associated with degenerate eigenvalues can be found, which is a natural extension to the method of Nelson and finalizes that of Ojalvo. This is summarized as follows; derivations may be found thereof. Define the right-hand side of Eq. (5) as

$$F \equiv BZ\frac{\partial \boldsymbol{\Lambda}}{\partial r} - \left(\frac{\partial A}{\partial r} - \lambda\frac{\partial B}{\partial r}\right)Z \tag{14}$$

and the left-hand side as

$$G \equiv A - \lambda B \tag{15}$$

Note that the pencil G is of rank $(n-m)$ because λ is an m-fold degenerate eigenvalue of Eq. (1). This singularity of G is treated as follows to allow for its inversion and, hence, the particular solution for the eigenvector derivative from Eq. (5). If the lth element of $|Z_d|$ is $\|Z_d\|_\infty$, this may be achieved by letting F^* and G^* have all entries equal to F and G, respectively, with the exceptions

$$F^*_{l,d} = 0 \tag{16}$$

$$G^*_{l,j} = \delta_{l,j}, \qquad G^*_{i,l} = \delta_{i,l}, \qquad i, j = 1, \ldots, n \tag{17}$$

for $c = 1, \ldots, m$, without repetition of l. This latter stipulation may demand that the lth element not be the maximum element, but instead a successive maximum. With the singularity of G overcome, the particular solution to the eigenvector derivative is then, directly from Eq. (5),

$$V = (G^*)^{-1} F^* \tag{18}$$

Differentiation of the orthonormality condition (9) leads to the definition

$$Q = -\left(V^T B Z + Z^T \frac{\partial B}{\partial r} Z + Z^T B V\right) \tag{19}$$

while differentiation of Eq. (5) leads to the definition

$$R = Z^T\left(\frac{\partial A}{\partial r} - \lambda\frac{\partial B}{\partial r}\right)V - Z^T\left(\frac{\partial B}{\partial r}Z + BV\right)\frac{\partial \boldsymbol{\Lambda}}{\partial r} + \frac{1}{2}Z^T\left(\frac{\partial^2 A}{\partial r^2} - \lambda\frac{\partial^2 B}{\partial r^2}\right)Z \tag{20}$$

The complementary function C provides the constraint for the general solution, completing the eigenvector derivative

$$\frac{\partial Z}{\partial r} = V + ZC \tag{21}$$

The constraint depends on the type of degeneracy because away from the point at which the derivative is defined the transitory and permanent types specify different conditions. Use is then made of the definitions (19) and (20). If degeneracy is first-order transitory, then

$$C_{ij} = R_{ij}\left(\frac{\partial \lambda_j}{\partial r} - \frac{\partial \lambda_i}{\partial r}\right)^{-1} \tag{22}$$

if it is permanent,

$$C_{ij} = \tfrac{1}{2} Q_{ij} \tag{23}$$

In the permanently degenerate case, the solution to Eq. (21) with C defined in Eq. (23) is a nonunique derivative that is directionally admissible. For differentiability in this direction to be defined, all bases must be aligned by Eq. (13). If there is no degeneracy, then the eigenvalues are distinct and Nelson's method is applicable, which is analogous to that given earlier with the complementary function

$$c = -v^T B x - \frac{1}{2}\left(x^T \frac{\partial B}{\partial r} x\right) \tag{24}$$

where v is the particular solution to

$$\frac{\partial x}{\partial r} = v + cx \tag{25}$$

Mode Traces

The philosophy of mode tracing may be expressed as follows: Let there be $1, \ldots, s$ modes in the considered numerical iteration subset S at iterate k and let there be $1, \ldots, t$ modes in a subset T at iterate $(k+1)$, following a perturbation. Generally, t should be greater than s so that T encompass all of the modes in S. Furthermore, let

$$M_{ij}\left(z_i^k, \overline{z_j^{k+1}}\right) \in [0, 1], \qquad i = 1, 2, \ldots, s, \qquad j = 1, 2, \ldots, t \tag{26}$$

be the elements of an $s \times t$ matrix connoting some normalized quantifier for vector consistency between the ith eigenvector in S and the jth eigenvector in T. If the sets exist in t-column matrices, the correcting permutation of modes in T may be expressed as

$$\left\{\overline{z_1^{k+1}} \cdots \overline{z_t^{k+1}}\right\} \rightarrow \left\{\overline{z_1^{k+1}} \cdots \overline{z_t^{k+1}}\right\}\boldsymbol{\Theta}^T \tag{27}$$

and, consistently,

$$\left\{\lambda_1^{k+1} \cdots \lambda_t^{k+1}\right\} \rightarrow \left\{\lambda_1^{k+1} \cdots \lambda_t^{k+1}\right\}\boldsymbol{\Theta}^T \tag{28}$$

where the $s \times t$ Boolean operator $\boldsymbol{\Theta}$ is the binary matrix whose rows contain exclusive units corresponding to the row maxima of $\boldsymbol{M}$. By Eqs. (27) and (28), with the assumption that there is no ambiguity in vector correlation, there are always s modes with physically consistent permutations in consideration.

Augmented Modal Assurance Routine

Ambiguity in modal correlation may be introduced if the perturbation to r is large, due to significant variations in the eigenvectors. Consequently, the exclusive row units of $\boldsymbol{\Theta}$ cannot be designated with confidence due to this ambiguity in the consistency matrix. An incorrectly developed binary operator will cause a relapse into the original problem of discontinuous eigenvalue functions. The following additions to the vector consistency evaluation are, therefore, proposed to alleviate potential ambiguity:

$$\overrightarrow{\boldsymbol{M}}_{ij}\left(\left[z_i^k + \frac{\partial z_i^k}{\partial r}(r^{k+1} - r^k)\right]w^{-\frac{1}{2}}, \overline{z_j^{k+1}}\right) \in [0, 1] \qquad (29)$$

$$\overleftarrow{\boldsymbol{M}}_{ij}\left(z_i^k, \left[\overline{z_j^{k+1}} + \frac{\partial \overline{z_j^{k+1}}}{\partial r}(r^k - r^{k+1})\right]w^{-\frac{1}{2}}\right) \in [0, 1] \qquad (30)$$

where $i = 1, 2, \ldots, s$, $j = 1, 2, \ldots t$, and w is a scalar defined by Eq. (8) with $\boldsymbol{\Phi}$ equal to the vector w multiplies. $\boldsymbol{B}$-orthonormality must be enforced for XOR to have normalized values, but is not necessary for MAC, which is independent of $\boldsymbol{B}$ and has a normalizing denominator. The definitions of these consistency evaluations can be found in Eqs. (31) and (32). Equations (29) and (30) supplement Eq. (26) by making affine models of the freedoms of the eigenvectors at the kth and $(k+1)$th iterates, respectively, from knowledge of their derivatives and the amount by which the parameter has been perturbed by an iterative excursion. Thus, the insight into vector consistency across parameter intervals may be augmented threefold. Depending on the degree of degeneracy, Eqs. (21–25) give the eigenvector derivatives featured in Eqs. (29) and (30). In all permanently degenerate cases, the eigenvector bases of the proceeding iterate should be aligned to those at the previous iterate by Eq. (13). Naturally, performance of this alignment is hindered by ignorance of which modes are to be reconciled across a parameter interval, this, of course, being the end of the mode tracing effort. The resolution is to perform all of the possible alignments that result in a number of consistency matrices (26), (29), or (30), equal to the number of degenerate modes in set T, t. Accordingly, the Boolean operator of Eqs. (27) and (28) has, in that instance, row units corresponding then to the global row maxima of the set of consistency matrices.

Numerical Example

The following example serves to demonstrate mode traces in degenerate eigensystems and the augmentation to the assurance by which this tracing can be performed. Figure 3 shows a cyclic structure formed by a pentagon of unit circumradius accommodating a pentalpha in its interior. At each intersection of members are two translational freedoms, so that the entire structure is associated with 10 freedoms. Table 1 gives the geometric and material properties nominated for the members. Although the problem is not physically realizable, it serves to demonstrate the concepts outlined here and the utility of the proposed augmented assurance.

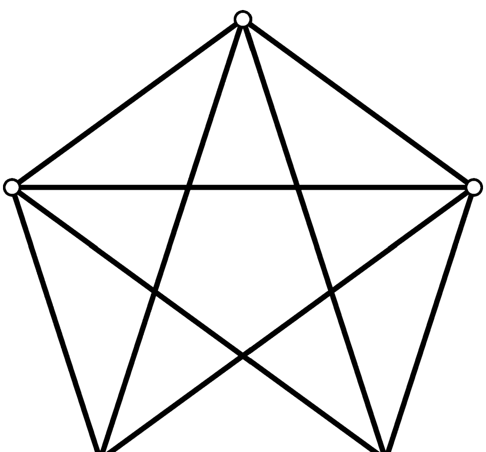

Fig. 3 Pentagonal structure.

Table 1 Material and geometrical properties

Parameter	Value
Young's modulus, E, $\mathrm{N \cdot m^{-2}}$	2.10×10^{11}
Density ρ, $\mathrm{kg \cdot m^{-3}}$	7.85×10^3
Diameter ϕ, m	1.00×10^{-2}
Area A, $\mathrm{m^2}$	$\pi \times 10^{-4}$

Table 2 Statically admissible forces

Member	Value
Pentagonal	$+(8/5)r$
Pentalpha	$-(8/10)\sec(\pi/5)r$

The perturbation to the eigenpairs of the structural eigenproblem is chosen to be induced by a cyclically symmetric distribution of membrane forces within the frame. The resulting conservation of symmetry with respect to parameter variations entails permanent degeneracy of certain eigenvalues. The statically admissible force distribution is shown in Table 2, where r is defined. (Note that r is quoted in Newtons, although it is a quantifier for the force distribution and not the force in any particular member.) For discussion of the finite element analysis of the modes of a specific plane frame experiencing membrane forces, the reader is referred to Greening and Lieven[15]; a rigorous, analytical study of the same frame is given by Mead.[16]

The stiffness and mass matrices of the members of the pentagonal structure are taken, respectively, to be

$$\boldsymbol{A}^e = [\boldsymbol{T}_1]^T \frac{1}{L}\begin{bmatrix} EA & 0 & -EA & 0 \\ 0 & r & 0 & -r \\ -EA & 0 & EA & 0 \\ 0 & -r & 0 & r \end{bmatrix}[\boldsymbol{T}_1]$$

$$\boldsymbol{B}^e = [\boldsymbol{T}_2]^T \frac{\rho A L}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}[\boldsymbol{T}_2]$$

where L is the member length and $[\boldsymbol{T}_1]$ and $[\boldsymbol{T}_2]$ are, respectively, the 4×4 and 2×4 coordinate transformation matrices; other parameters are defined in Table 1. The stiffness matrix derivative is then

$$\frac{\partial \boldsymbol{A}^e}{\partial r} = [\boldsymbol{T}_1]^T \frac{1}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}[\boldsymbol{T}_1]$$

The mass matrix is, of course, stationary with respect to the membrane forces, and its derivative is zero. The member stiffness and mass matrices are assembled into respective global matrices to form the eigenproblem (1) of order 10. The zero-frequency rigid-body modes, which here amount to three, are excluded in subsequent analysis, and, hence, the initial mode designation relates to the initial membrane mode. It follows that there are in total seven modes under consideration. The variations in eigenvalues are shown in Fig. 4 for the range from $r = 0$ to $r = 1 \times 10^8$ N. Note that the forces required to vary the eigenvalues discernibly are of considerable magnitude. This is the result of the high energy associated with membrane modes, which are the class of all of the strain modes exhibited by the present structure incapable of flexure.

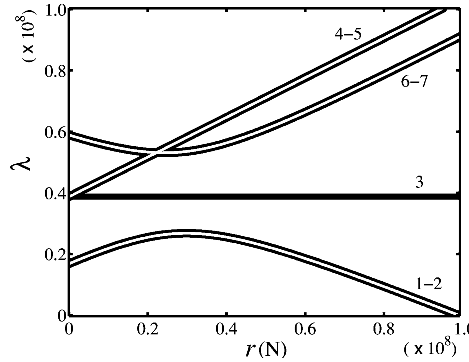

Fig. 4 Eigenvalue functions of *r* (inlaid curves indicative of permanently degenerate pairs).

The modal designations are given notwithstanding the coalescence and are indicative of the ordering at $r = 0$ N. Mode 1–2 is seen to reduce to zero in this range and exhibits single curvature, as does mode 6–7; mode 4–5 is linear, and mode 3 is completely unaffected. Mode 4–5 is analogous to the modes of a single bar, whose normalized mode shapes are stationary or near stationary with respect to its axial force, whereas the eigenvalues are linear or quasi-linear functions of that force. The curvatures in other modes can be explained by the coupling of two linear portions in which the curvature connotes modal smearing and, hence, the transformation of the eigenvectors of one branch into those of the other. In such a region, the rapidly transforming eigenvectors can become unrecognizable from the viewpoint of either branch. It is, however, the case that at these regions, the eigenvectors contain components of the vectors associated with each of the pair of branches the regions connect. Presently, the intention is that these components can be surmised from the forward and backward casting of vector freedoms, making the coupled eigenvectors recognizable to each branch.

It is required to trace modes across the interval from $r = 0$ to $r = 3 \times 10^7$ N, with the latter point coinciding with the vicinity of the stationarities of modes 1–2 and 6–7. Here, the eigenvectors are partially representative of the respective pairs of branches of modes 1–2 and 6–7 and, hence, signify deviations from their forms at $r = 0$ N in the leftmost branches. Reference 2 treats such a coupling problem by suspending tracing of an affected mode. Whereas this works in the case of one coupling event, tracing may be made impossible if multiple events of coupling demand that a set of modes be excluded from a trace. This may be understood by recalling the aforementioned concept that the eigenvectors in a coupling transition may be unrecognizable to a preceding iterate. If this is so, and the current iterate is near the stationarity of the eigenvalue, then it is unlikely that these vectors will be recognizable to a proceeding iterate. It then follows that, although one excluded mode may be reinstated in subsequent sets by inference, this may not hold for multiple exclusions. Hence, the insight offered by forward and backward casting of eigenvectors is seen as an especial aid in this context. Such is the case in the present example. Fig. 4 illustrates the two eigenvalue stationarities nearly coincident at a particular parameter value.

The measures of vector consistency considered here are MAC and XOR, which are defined for the sets of seven orthonormal vectors as

$$\mathrm{MAC}_{ij}\left(z_i^k, \overline{z_j^{k+1}}\right) = \frac{\left|\left(z_i^k\right)^T\left(\overline{z_j^{k+1}}\right)\right|^2}{\left\|z_i^k\right\|_2^2\left\|\overline{z_j^{k+1}}\right\|_2^2} \in [0, 1] \tag{31}$$

$$\mathrm{XOR}_{ij}\left(z_i^k, \overline{z_j^{k+1}}\right) = \left|\left(z_i^k\right)^T \boldsymbol{B}\left(\overline{z_j^{k+1}}\right)\right| \in [0, 1] \tag{32}$$

The results of employing Eqs. (31) and (32) to obtain eigenvector consistencies between $r = 0$ and $r = 3 \times 10^7$ N are given in the following matrices to three decimal places:

$$\mathrm{MAC} = \begin{bmatrix} 0.617 & 0 & 0 & 0 & 0.088 & 0 & 0 \\ 0 & 0.617 & 0 & 0.088 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0.002 & 0.447 & 0 & 0.870 & 0 & 0 & 0 \\ 0.447 & 0.002 & 0 & 0 & 0.870 & 0 & 0 \end{bmatrix}$$

$$\mathrm{XOR} = \begin{bmatrix} 0.872 & 0 & 0 & 0.036 & 0.488 & 0 & 0 \\ 0 & 0.872 & 0 & 0.488 & 0.036 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0.036 & 0.488 & 0 & 0.872 & 0 & 0 & 0 \\ 0.488 & 0.036 & 0 & 0 & 0.872 & 0 & 0 \end{bmatrix}$$

The linear eigenvalue functions, whose eigenvectors are stationary, expectedly result in unit consistencies. With MAC, the remaining modes are correlated with wanting assurance. Although this assurance is greater with XOR, there exist in both considerable corruption terms.

The consistency evaluation (29) is now employed on the inadequately traced modes, in which the eigenvectors at $r = 0$ N are forward cast so that they might augment the assurance of mode tracing. Note that these eigenvectors are associated with modes 1–2 and 6–7, which are permanently degenerate pairs. The forward casting requires computation of the eigenpair derivatives. The eigenvalue derivatives and the transformation matrices orthogonalizing and aligning the eigenvectors across the parameter interval are evaluated from Eqs. (6) and (11). The eigenvector derivatives are then evaluated by Eqs. (21) and (23). These eigenpair derivatives are shown in Table 3, along with the respective eigenvalues, to three significant figures. The resulting forward cast eigenvectors are shown in comparison to their original forms at $r = 0$ N and to those at $r = 3 \times 10^7$ N in Tables 4–6; all vector freedoms are quoted to three significant figures. The anticipation is that the resemblance between the eigenvectors at $r = 3 \times 10^7$ N and the forward cast vectors is greater than with the eigenvectors at $r = 0$ N, and such is the indication. Note that not all of the freedoms in Table 6 approximate those in Table 5. This is because of the shortfall of the affine modeling of the freedoms. However, if the assumption of affineness holds for enough freedoms, as is likely, then the forward cast vectors will lead to greater consistency.

The $\overrightarrow{\mathrm{MAC}}$ and $\overrightarrow{\mathrm{XOR}}$ matrices evaluated across the interval from $r = 0$ N to $r = 3 \times 10^7$ N, following the forward casting of vectors, are shown hereafter with values stated to three decimal places. In

Table 3 Eigenpair derivatives at $r = 0$ N

Mode 1	Mode 2	Mode 6	Mode 7
	λ		
1.68×10^7	1.68×10^7	5.89×10^7	5.89×10^7
	$\partial\lambda/\partial r$		
0.546	0.546	−0.393	−0.393
	$\partial \mathbf{z}/\partial r$		
2.18×10^{-9}	2.09×10^{-9}	-1.13×10^{-9}	-9.69×10^{-10}
1.40×10^{-9}	1.69×10^{-9}	1.15×10^{-9}	1.66×10^{-9}
-2.95×10^{-9}	-6.42×10^{-10}	-1.36×10^{-9}	-6.00×10^{-10}
2.19×10^{-9}	2.19×10^{-10}	-2.00×10^{-9}	-3.23×10^{-10}
-9.50×10^{-10}	6.78×10^{-10}	2.17×10^{-9}	-9.65×10^{-10}
-3.21×10^{-9}	1.50×10^{-9}	-8.22×10^{-10}	1.61×10^{-11}
1.75×10^{-9}	-3.29×10^{-9}	2.57×10^{-10}	1.62×10^{-10}
1.80×10^{-10}	9.56×10^{-11}	1.33×10^{-9}	-2.11×10^{-9}
-3.19×10^{-11}	1.17×10^{-9}	6.03×10^{-11}	2.37×10^{-9}
-5.54×10^{-10}	-3.50×10^{-9}	3.39×10^{-10}	7.49×10^{-10}

Table 4 Eigenvectors at $r = 0$ N

Mode 1	Mode 2	Mode 6	Mode 7
−0.0975	0.132	0.248	−0.223
0.174	−0.140	0.198	−0.141
−0.0549	0.155	−0.0949	0.319
−0.0193	0.223	0.0420	−0.239
−0.124	−0.231	0.0670	0.110
0.00851	0.0905	0.138	0.366
0.0158	−0.0297	−0.348	−0.220
−0.243	−0.129	0.0120	−0.0190
0.261	−0.0261	0.128	0.0131
0.0798	−0.0435	−0.390	0.0324

Table 5 Eigenvectors at $r = 3 \times 10^7$ N

Mode 1	Mode 2	Mode 6	Mode 7
0.0328	0.228	0.155	−0.247
0.228	−0.0310	0.235	−0.0326
−0.207	0.100	−0.156	0.246
0.101	0.206	−0.0714	−0.226
−0.160	−0.165	0.176	0.0439
−0.166	0.160	0.0763	0.320
0.108	−0.204	−0.290	−0.183
−0.202	−0.108	0.0824	−0.131
0.226	0.0404	0.115	0.140
0.0396	−0.227	−0.322	0.0688

Table 6 Forward cast vectors from $r = 0$ N

Mode 1	Mode 2	Mode 6	Mode 7
−0.0310	0.188	0.206	−0.243
0.208	−0.0866	0.224	−0.0876
−0.138	0.131	−0.131	0.291
0.0447	0.221	−0.0173	−0.240
−0.147	−0.203	0.127	0.0785
0.0848	0.131	0.110	0.354
0.0659	−0.124	−0.328	−0.207
−0.229	−0.122	0.0501	−0.0793
0.251	0.00864	0.126	0.0812
0.0610	−0.143	−0.367	0.0530

both matrices, assurance is augmented and corruption greatly diminished. Thus, mode tracing may be executed with heightened confidence:

$$\overrightarrow{\mathrm{MAC}} = \begin{matrix} 1 \\ 2 \\ 6 \\ 7 \end{matrix}\begin{bmatrix} 0.916 & 0 & 0 & 0.005 \\ 0 & 0.916 & 0.005 & 0 \\ 0.002 & 0.274 & 0.966 & 0 \\ 0.274 & 0.002 & 0 & 0.966 \end{bmatrix}$$
$$\qquad\qquad 1 \qquad 2 \qquad 6 \qquad 7$$

$$\overrightarrow{\mathrm{XOR}} = \begin{matrix} 1 \\ 2 \\ 6 \\ 7 \end{matrix}\begin{bmatrix} 0.970 & 0 & 0.018 & 0.243 \\ 0 & 0.970 & 0.243 & 0.018 \\ 0.018 & 0.243 & 0.970 & 0 \\ 0.243 & 0.018 & 0 & 0.970 \end{bmatrix}$$
$$\qquad\qquad 1 \qquad 2 \qquad 6 \qquad 7$$

One further point that needs to be raised is that of mode veering. In the present example, this phenomenon is exhibited by modes 1–2 and 6–7 and can be thought of as a deviation from the coalescence case in that the eigenvalues do not intersect but veer away from one another. Thus it is in the limiting case that the eigenvalues intersect; both mode veering and eigenvalue coalescence are then manifestations of the same phenomenon. A detailed discussion of these two features of eigenvalue curves is given by Liu[17]; mode veering is experienced in the study by Mead[16] in which the analytical results indicate narrow ranges of abrupt changes in mode shapes.

The immediate repercussions of mode veering are as follow. First, the aforementioned possibility of the coincidence of more than one coupling event is in fact probable and therefore suspension of the tracing of the modes in such events is not suitable; the proposed augmented assurance is here an especial aid. Second, if coalescence is thought of as the limiting case of mode veering, then opposite branches of two veering curves possess similar eigenvectors. In Fig. 4, the left branch of modes 1–2 has eigenvectors similar to those in the right branch of modes 6–7 and vice versa. Indeed, this is an explanation for the corruption terms associated with modes 1–2 and 6–7 in the original consistency matrices. Again, the augmented assurance helps to lessen ambiguity. It is apparent that there must exist situations in which mode tracing leads to the impression that two opposite branches of two separate eigenvalue curves are a single curve. However, if the veering phenomenon is thought of as a deviation from the case of coalescence, then there is conservation of function continuity, albeit in a loose sense, and, therefore, no hindrance is presented to gradient-based iterative methods. There is, of course, the stipulation that if two branches of separate curves are traced as a single curve, then the remaining two must themselves be traced to be consistent, but both the former and the latter are likely to be mutually inclusive events.

Conclusions

Mode tracing in eigensystems that exhibit eigenvalues of varying degrees of degeneracy has been outlined. Emphasis is placed on the alignment of eigenvector bases to enable continuity and, therefore, differentiability of eigenvectors to be observed. This alignment needs extensive treatment in the permanently degenerate case, with the consideration of a specific, basis-aligning transformation matrix that is not required in the other cases.

A supplement to vector consistency evaluation that potentially augments the assurance with which modes can be traced has been proposed; note that a threefold increase to the insight of modal correlation can be gained. It remains to develop an automated strategy that utilizes information from all three consistency matrices. In the permanently degenerate case, all of the respective possible matrices of the three consistency matrices, arising from the various combinations of vector alignment, need to be processed before their information can be utilized.

A numerical example has served to demonstrate that the proposal overcomes a deficiency in eigenvector correlation associated with tracing modes away from and at an event of mode coupling, where eigenvectors exhibit rapid variations. Certainly, eigenvalue gradients should be used when possible as indicators of the coupling region in the vicinity of eigenvalue stationarity, but the potentially augmented assurance may make unnecessary the need to exclude modes temporarily in the mode tracing task. Furthermore, the proposal holds the potential of increasing the permissible parameter perturbations that iterative schemes dictate in terms of the assurance with which modes can be traced.

The phenomenon of mode veering, its relation to eigenvalue coalescence, and its repercussions to mode traces is discussed. Incorrect mode traces, in the sense of traces traversing modal curves, are deemed permissible if function continuity is conserved.

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